Q1

1

The points $A(2,-21)$ and $B(-5,3)$ are the two endpoints of the diameter AB of a circle. Find the equation of the circle in the form $ax^2 + ay^2 + bx + cy + d = 0$, where a, b, c and d are integers to be found.

[6]

CENTRE = MIDPOINT AB

$C = \left(\frac{2-5}{2}, \frac{-21+3}{2}\right) = \left(-\frac{3}{2}, -9\right)$

$(x-a)^2 + (y-b)^2 = r^2$

$\left(x+\frac{3}{2}\right)^2 + (y+9)^2 = r^2$

$A(2,-21)$ $\quad \left(2+\frac{3}{2}\right)^2 + (-21+9)^2 = r^2$

$r^2 = \left(\frac{7}{2}\right)^2 + (-12)^2 = \frac{625}{4}$

$\left(x+\frac{3}{2}\right)^2 + (y+9)^2 = \frac{625}{4}$

EXPAND

$x^2 + 3x + \frac{9}{4} + y^2 + 18y + 81 = \frac{625}{4}$

$\times 4$ $\quad \times 4$

$4x^2 + 12x + 9 + 4y^2 + 72y + 324 = 625$

$\boxed{4x^2 + 4y^2 + 12x + 72y - 292 = 0}$

Q2

2

Find the centre and radius of the circle with equation $x^2 + y^2 + x - 3y + 2 = 0$.

[4]

$(x-a)^2 + (y-b)^2 = r^2$

COMPLETE THE SQUARE

$x^2 + x + y^2 - 3y + 2 = 0$

$\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + \left(y-\frac{3}{2}\right)^2 - \frac{9}{4} + 2 = 0$

$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{1}{2}$

CENTRE $= \left(-\frac{1}{2}, \frac{3}{2}\right)$ $\quad r = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$\boxed{C = -\frac{1}{2}, \frac{3}{2} \qquad r = \frac{\sqrt{2}}{2}}$

Q3

3

The line $x + y = c$ intersects the circle $x^2 + y^2 - 6x + 10y - 16 = 0$ at exactly two points. Find the range of possible values of c.

[7]

REARRANGE ① $y = c - x$

SUB INTO ②

$x^2 + (c-x)^2 - 6x + 10(c-x) - 16 = 0$

$x^2 + c^2 - 2cx + x^2 - 6x + 10c - 10x - 16 = 0$

$2x^2 - 2cx - 16x + c^2 + 10c - 16 = 0$

CONSIDER DISCRIMINANT FOR EXACTLY TWO SOLUTIONS $b^2 - 4ac > 0$

$a = 2 \quad b = -2c - 16 \quad c = c^2 + 10c - 16$

$(-2c-16)^2 - 4(2)(c^2 + 10c - 16) > 0$

$4c^2 + 64c + 256 - 8c^2 - 80c + 128 > 0$

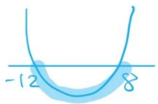
$-4c^2 - 16c + 384 > 0$

$0 > 4c^2 + 16c - 384 \quad \div 4$

$\div 4 \quad 0 > c^2 + 4c - 96$

$0 > (c-8)(c+12)$

$c = 8 \quad c = -12$



$\boxed{-12 < c < 8}$

Q4

4

The points $A(-2, 3)$, $B(0, 6)$ and $C(k, -1)$ lie on a circle, where BC is the diameter of the circle.

Find the value of k.

[4]

USING ANGLE IN SEMICIRCLE PROPERTY

BC = DIAMETER

THEREFORE MUST BE HYPOTENUSE OF RIGHTANGLED TRIANGLE

$a^2 + b^2 = c^2$

$AB^2 + AC^2 = BC^2$

$AB^2 = (0+2)^2 + (6-3)^2$

$2^2 + 3^2 = 13$

$AC^2 = (k+2)^2 + (-1-3)^2$

$k^2 + 4k + 4 + (-4)^2$

$k^2 + 4k + 20$

$BC^2 = k^2 + (-1-6)^2$

$k^2 + 49$

$AB^2 + AC^2 = BC^2$

$13 + \cancel{k^2} + 4k + 20 = \cancel{k^2} + 49$

$-33 \quad 4k = 16 \quad -33$

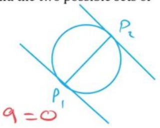
$\boxed{k = 4}$

Q5

5

A circle C has equation $x^2 + y^2 - 10x - 4y + 19 = 0$. Point P lies on the circle, and the tangent to the circle at point P has a gradient of -3. Find the two possible sets of coordinates for point P.

[7]



COMPLETE THE SQUARE

$x^2 - 10x + y^2 - 4y + 19 = 0$

$(x-5)^2 - 25 + (y-2)^2 - 4 + 19 = 0$

$(x-5)^2 + (y-2)^2 = 10$ ②

CENTRE $= (5, 2)$

EQUATION OF DIAMETER PERPENDICULAR TO TANGENTS

$m = \frac{1}{3}$ CENTRE $= (5,2)$

$y - 2 = \frac{1}{3}(x-5)$

$y - 2 = \frac{1}{3}x - \frac{5}{3}$ ①

$y = \frac{1}{3}x + \frac{1}{3}$

POINTS OF INTERSECTION SUB ① INTO ②

$(x-5)^2 + \left(\frac{1}{3}x - \frac{5}{3}\right)^2 = 10$

$x^2 - 10x + 25 + \frac{1}{9}x^2 - \frac{10}{9}x + \frac{25}{9} - 10 = 0$ $\times 9$

$\times 9$

$9x^2 - 90x + 225 + x^2 - 10x + 25 - 90 = 0$

$10x^2 - 100x + 160 = 0$ $\div 10$

$\div 10$ $x^2 - 10x + 16 = 0$

$(x-2)(x-8)$ $x = 2$ $x = 8$

SUB INTO $y = \frac{1}{3}x + \frac{1}{3}$

$x = 2$ $y = \frac{1}{3}(2) + \frac{1}{3} = 1$ $(2, 1)$

$x = 8$ $y = \frac{1}{3}(8) + \frac{1}{3} = 3$ $(8, 3)$

$\boxed{P_1 = (2,1) \quad P_2 = (8,3)}$

Q6

6

The points $A(4, 6)$, $B(7, 2)$ and $C(12, 12)$ lie on a circle.

Find the equation of the circle. $(x-a)^2 + (y-b)^2 = r^2$

[7]

USING PYTHAGORAS AND ANGLE IN SEMICIRCLE PROPERTY TO FIND DIAMETER

$a^2 + b^2 = c^2$

$AB^2 = (7-4)^2 + (2-6)^2 = 3^2 + (-4)^2 = 25$

$BC^2 = (12-7)^2 + (12-2)^2 = 5^2 + 10^2 = 125$

$AC^2 = (12-4)^2 + (12-6)^2 = 8^2 + 6^2 = 100$

$AB^2 + AC^2 = BC^2$

BC = HYPOTENUSE = DIAMETER

CENTRE = MIDPOINT BC (a, b)

$C = \left(\frac{7+12}{2}, \frac{2+12}{2}\right) = \left(\frac{19}{2}, 7\right)$

r = HALF DIAMETER BC

$r = \frac{BC}{2}$ $r^2 = \frac{BC^2}{2^2} = \frac{125}{4}$

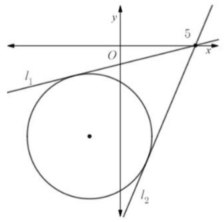
$\boxed{\left(x - \frac{19}{2}\right)^2 + (y-7)^2 = \frac{125}{4}}$

Q7

7

A circle has equation $x^2 + y^2 + 4x + 12y = -23$.

The lines l_1 and l_2 are both tangents to the circle, and they intersect at the point $(5, 0)$.



Find the equations of l_1 and l_2, giving your answers in the form $y = mx + c$. [8]

USE (5,0) TO FORM EQUATION FOR l_1 AND l_2

$y - 0 = m(x-5)$

$\frac{1}{m}y = x - 5$

$x = \frac{1}{m}y + 5$

SUB INTO CIRCLE EQUATION

$\left(\frac{1}{m}y+5\right)^2 + y^2 + 4\left(\frac{1}{m}y+5\right) + 12y = -23$

$\frac{1}{m^2}y^2 + \frac{10}{m}y + 25 + y^2 + \frac{4}{m}y + 20 + 12y + 23 = 0$

$\left(\frac{1}{m^2}+1\right)y^2 + \left(\frac{14}{m}+12\right)y + 68 = 0$

SINGLE SOLUTION FOR EACH LINE

DISCRIMINANT $= 0$ $\quad b^2 - 4ac = 0$

$a = \frac{1}{m^2}+1 \quad b = \frac{14}{m}+12 \quad c = 68$

$\left(\frac{14}{m}+12\right)^2 - 4\left(\frac{1}{m^2}+1\right)68 = 0$

$\frac{196}{m^2} + \frac{336}{m} + 144 - \frac{272}{m^2} - 272 = 0$

$-\frac{76}{m^2} + \frac{336}{m} - 128 = 0$

$\frac{76}{m^2} - \frac{336}{m} + 128 = 0$

$\frac{76}{m^2} - \frac{336}{m} + 128 = 0$

$f(x) = \frac{1}{m}$

$76x^2 - 336x + 128 = 0$

$x = 4 \qquad x = \frac{8}{19}$

$x = \frac{1}{m}$

$\frac{1}{m} = 4 \qquad \frac{1}{m} = \frac{8}{19}$

$l_1 = m = \frac{1}{4} \qquad l_2 = m = \frac{19}{8}$

USING m AND (5,0)

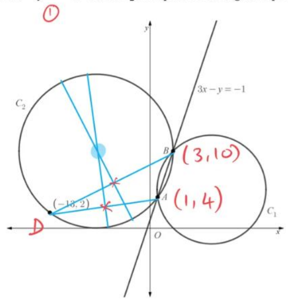
$y = \frac{1}{4}(x-5) \qquad y = \frac{19}{8}(x-5)$

$\boxed{l_1 \; y = \frac{1}{4}x - \frac{5}{4} \qquad l_2 \; y = \frac{19}{8}x - \frac{95}{8}}$

Q8

8

The diagram below shows circles C_1 and C_2 which intersect at the two points A and B.
② Circle C_1 has equation $x^2 + y^2 - 16x - 10y + 39 = 0$, and points A and B lie along the line with equation $3x - y = -1$. Circle C_2 also passes through the point $(-13, 2)$.
①



Find an equation of circle C_2.

SOLVE SIMULTANEOUS EQUATIONS TO FIND A AND B

[11]

REARRANGE ① $\quad y = 3x + 1$

SUB INTO ②

$x^2 + (3x+1)^2 - 16x - 10(3x+1) + 39 = 0$

$x^2 + 9x^2 + 6x + 1 - 16x - 30x - 10 + 39 = 0$

$10x^2 - 40x + 30 = 0 \quad \div 10$

$\div 10$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3 \qquad x = 1$

SUB INTO ①

$x = 3 \qquad 3(3) - y = -1 \qquad y = 10$

$9 - y = -1$

$x = 1 \qquad 3(1) - y = -1 \qquad y = 4$

$3 - y = -1$

$A = (1, 4) \qquad B = (3, 10)$

FIND EQUATIONS OF PERPENDICULAR BISECTORS USING $m = -\dfrac{x_2 - x_1}{y_2 - y_1}$

AD

MIDPOINT $\left(\dfrac{1-13}{2}, \dfrac{4+2}{2}\right) = (-6, 3)$

AD

MIDPOINT $\left(\dfrac{1-13}{2}, \dfrac{4+2}{2}\right) = (-6, 3)$

$M_1 = -\dfrac{1+13}{4-2} = -7$

EQUATION OF PERP BISECTOR TO AD

$y - 3 = -7(x+6)$

$y = -7x - 39$

BD

MIDPOINT $\left(\dfrac{3-13}{2}, \dfrac{10+2}{2}\right) = (-5, 6)$

$M_2 = -\dfrac{3+13}{10-2} = -2$

EQUATION OF PERP BISECTOR TO BD

$y - 6 = -2(x+5)$

$y = -2x - 4$

CENTRE OF CIRCLE WHERE BISECTORS INTERSECT

$-7x - 39 = -2x - 4$

$5x = -35$

$x = -7$

SUB TO FIND y

$y = -2(-7) - 4$

$y = 10$

$x = -7$ $\qquad y = 10$

CENTRE $(-7, 10)$

$$(x+7)^2 + (y-10)^2 = r^2$$

SUB IN $(-13, 2)$ TO FIND r^2

$$(-13+7)^2 + (2-10)^2 = r^2$$

$$(-6)^2 + (-8)^2 = 100 \qquad r^2 = 100$$

EQUATION OF C_2

$$\boxed{(x+7)^2 + (y-10)^2 = 100 \quad \text{or} \quad x^2 + y^2 + 14x - 20y + 49 = 0}$$